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RESEARCH CONNECTION

On the generalized fractional Laplacian

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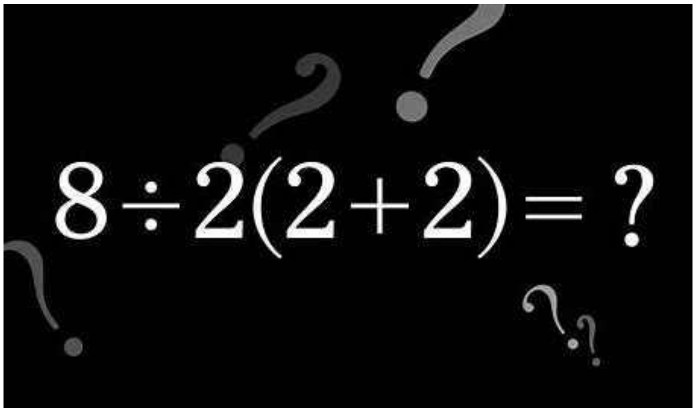

What you need to know

Fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the integer order differentiation and n-fold integration. The beginning of the fractional calculus is considered to be Leibniz's letter to L'Hôpital dated September 30, 1695, which discussed the notation for differentiation of non-integer order 1/2. The main advantage of fractional calculus is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Why this research is important

During the past few decades, fractional calculus has been a useful tool for developing more sophisticated mathematical models that can accurately describe complex systems. Fractional powers of the Laplacian operator arise naturally in the study of anomalous diffusion, where the fractional operator plays an analogous role to that of the integer order Laplacian for ordinary diffusion. By replacing the Brownian motion of particles with Lévy flights, one obtains a fractional diffusion equation (or fractional kinetic equation) in terms of the fractional Laplacian operator of order $0 < s < 1$ via the Cauchy principal value integral. The fractional Laplacian operator of order bigger than or equal to one remains undefined by an explicit integral formula, although there is strong demand in many applications.

How the research was conducted

Studying the fractional Laplacian through the use of distribution theory is a new approach that requires a mixture of knowledge in several fields with fresh ideas and innovative techniques. We have been working on this topic since 2018 and made the first connection between the fractional Laplacian and generalized functions based on Gel'fand normalization, Pizzetti's formula, and surface integrals in an n-dimensional space to successfully extend the operator for all orders bigger than zero and even some complex numbers.

What the researchers found

For the first time, we have defined the fractional Laplacian over a new space $C_k(R^n)$, which contains $S(R^n)$ as a proper subspace, for any order bigger than zero. We further presented two theorems showing that our extended fractional Laplacian operator is continuous at the endpoints with examples opening a new direction for defining the fractional Laplacian for complex values by analytic continuation.

How this research can be used

The results obtained cannot be achieved in the classical (ordinary) sense and can be used to study many partial differential equations arising from different areas of science and engineering.
In July 2020, we established an explicit integral representation of the generalized Riesz derivative, which was widely considered to not exist in the past, and corrected an error that recently appeared in "Fractional Calculus and Applied Analysis."

About the researchers

Dr. Chenkuan Li is Professor in the Department of Mathematics & Computer Science at Brandon University. LiC@BrandonU.CA

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Joshua Beaudin is a Mathematics student at Brandon University and has worked with Dr. Chenkuan Li on fractional analysis in the summer of 2020 as a USRA.

Keywords

Fractional Laplacian; normalization; distribution; Pizzetti's formula; Gamma function; fractional calculus

Publications based on this research

Li, C. (2020). An example of the generalized fractional Laplacian. *Contemporary Mathematics, 1*(4) 215-226. https://doi.org/10.37256/cm.142020489

Li, C., & Beaudin, J. (2020). On the generalized Riesz derivative. *Mathematics, 8*(7), 1089; https://doi.org/10.3390/math8071089

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Li, C. & Huang, J. (2020).Remarks on the linear fractional integro-differential equation with variable coefficients in distribution. *Fractional Differential Calculus, 10*, 57–77.

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